Polarizability of the hydrogen atom

In order to calculate the polarizability of the hydrogen atom we need the transition moments for the allowed transitions. In the presence of a Z-polarized applied electric field we have the induced dipole moment

$$<\Psi_0|qZ|\Psi_0> = -2q^2E_0\sum_n \left(\frac{<n,l,m|Z|1,0,0>^2}{E_1 - E_n}\right)$$

Since $\mu = \alpha E_0$ we have for the polarizability

$$\alpha = -2q^2\sum_n \left(\frac{< n,l,m|Z|1,0,0 >^2}{E_1 - E_n}\right)$$

obtained from Cohen-Tannoudji (page 1280). It appears from the selection rules that Δn = any value, $\Delta l = \pm 1$, and $\Delta m = 0$.

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We have the following matrix elements:

$$<2,1,0|Z|1,0,0> = \int_0^\infty \frac{1}{\sqrt{32\pi}}\left(\frac{r}{a_0^{5/2}}\right)e^{-r/2a_0} r\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0} r^2 dr \int_0^\pi \cos^2\theta \sin\theta d\theta \int_0^{2\pi} d\phi$$

making the substitution x = cosθ, and collecting constants we have

$$<2,1,0|Z|1,0,0> = \frac{2\pi}{\sqrt{32}\pi}\left(\frac{1}{a_0^4}\right)\int_0^\infty e^{-3r/2a_0} r^4 dr \int_{-1}^{1} x^2 dx$$

Let u = $3r/2a_0$, then r = $(2a_0/3)$u and dr = $(2a_0/3)$du so

$$<2,1,0|Z|1,0,0> = \frac{64}{243\sqrt{32}} a_0 \int_0^\infty e^{-u} u^4 du \int_{-1}^{1} x^2 dx$$

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Evaluating the integrals we obtain

$$\langle 2,1,0|Z|1,0,0\rangle = \frac{64}{243\sqrt{32}} a_0 4! \frac{2}{3} = 0.744 a_0$$

The energy levels are given by

$$E_n = -\frac{\hbar^2}{2\mu a_0^2 n^2}$$

The energy difference in the denominator is

$$E_1 - E_n = \frac{\hbar^2}{2\mu a_0^2}\left(\frac{1}{n^2} - 1\right)$$

Polarizability of the hydrogen atom

The polarizability can be expressed

$$\alpha = \frac{2\mu q^2 a_0^{\,2}}{\hbar^2}\sum_n \left(\frac{< n,l,m|Z|1,0,0>^2}{1-\frac{1}{n}} \right)$$

Thus, for example for the 210 state we have

$$\alpha_{210} = \frac{2\mu q^2 a_0^{\,4}}{\hbar^2}\frac{0.744^2}{1-\frac{1}{4}} = \left(3.8\mathring{A}^3\right)0.738 = 2.80\ \mathring{A}^3$$

The constant in front of the polarizability expression is:

$$\frac{2\mu q^2 a_0^{\,4}}{\hbar^2} = \frac{2\left(9.109\times 10^{-31}kg\right)\left(1.62\times 10^{-19}C\right)^2\left(5.29\times 10^{-11}m\right)^4}{\left(1.054\times 10^{-34}Js\right)^2}$$

Polarizability of the hydrogen atom

The units work out as follows:

$$\frac{2\mu q^2 a_0{}^4}{\hbar^2} = 3.37 \times 10^{-41}\ C^2 m^2/J$$

$$\frac{2\mu q^2 a_0{}^4}{\varepsilon_0 \hbar^2} = \frac{3.37 \times 10^{-41} C^2 m^2/J}{8.854 \times 10^{-12} C^2 m^{-1}/J}$$

$$= 3.80 \times 10^{-30}\ m^3 = 3.80\ \mathring{A}^3$$