Spherical Polar Coordinates

We wish to transform from a three dimensional Cartesian coordinate system (x,y,z) to a spherical polar coordinate system (r,θ,ϕ). The relationship of the coordinate systems is:

$x = r \sin(\theta)\cos(\phi)$
$y = r \sin(\theta)\sin(\phi)$
$z = r \cos(\theta)$

as shown in Figure 1 below.

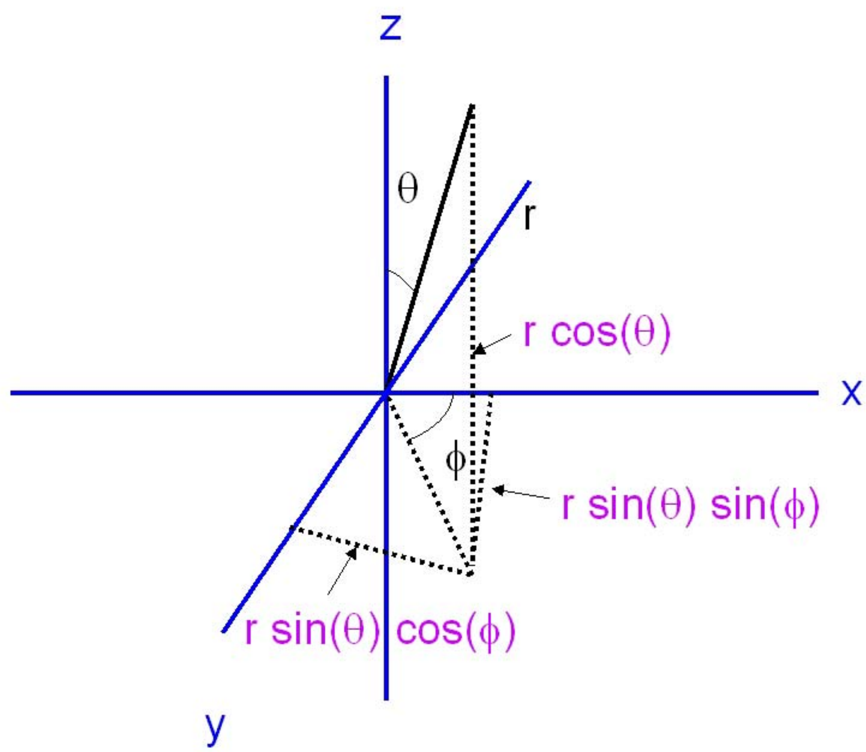

Figure 1. Definitions of the two coordinate systems.

A change variables involves defining x, y and z as a functions of r, θ and ϕ.

$x = f(r,\theta,\phi) = r \sin(\theta)\cos(\phi)$
$y = g(r,\theta,\phi) = r \sin(\theta)\sin(\phi)$
$z = h(r,\theta,\phi) = r \cos(\theta)$

Calculating the volume in Cartesian and spherical polar coordinates

The volume element in Cartesian coordinates is:

dV = dx dy dz

and the volume is calculated using a three-dimensional integral:

$$Volume = \iiint F(x,y,z)\,dxdydz$$

In polar coordinates there is a similar integral.

$$Volume = \iiint G(r,\theta,\phi)\ |J|\ dr\ d\theta\ d\phi$$

where J is the Jacobian, which is the determinant of the matrix that transforms from one coordinate system into the other:

$$dx = \frac{\partial f(r,\theta,\phi)}{\partial r}dr + \frac{\partial f(r,\theta,\phi)}{\partial \theta}d\theta + \frac{\partial f(r,\theta,\phi)}{\partial \phi}d\phi$$

$$dy = \frac{\partial g(r,\theta,\phi)}{\partial r}dr + \frac{\partial g(r,\theta,\phi)}{\partial \theta}d\theta + \frac{\partial g(r,\theta,\phi)}{\partial \phi}d\phi$$

$$dz = \frac{\partial h(r,\theta,\phi)}{\partial r}dr + \frac{\partial h(r,\theta,\phi)}{\partial \theta}d\theta + \frac{\partial h(r,\theta,\phi)}{\partial \phi}d\phi$$

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} \\ \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \theta} & \frac{\partial g}{\partial \phi} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \theta} & \frac{\partial h}{\partial \phi} \end{pmatrix}$$

$$\det \begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} \\ \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \theta} & \frac{\partial g}{\partial \phi} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \theta} & \frac{\partial h}{\partial \phi} \end{pmatrix} = \frac{\partial f}{\partial r} \begin{pmatrix} \frac{\partial g}{\partial \theta} & \frac{\partial g}{\partial \phi} \\ \frac{\partial h}{\partial \theta} & \frac{\partial h}{\partial \phi} \end{pmatrix} - \frac{\partial f}{\partial \theta} \begin{pmatrix} \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \phi} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \phi} \end{pmatrix} + \frac{\partial f}{\partial \phi} \begin{pmatrix} \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \theta} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \theta} \end{pmatrix}$$

To transform to polar coordinates we will need the following derivatives:

$$\left(\frac{\partial f}{\partial r}\right)_{\theta,\phi} = \sin(\theta)\cos(\phi)\ ,\ \left(\frac{\partial f}{\partial \theta}\right)_{r,\phi} = r\cos(\theta)\cos(\phi)\ ,\ \left(\frac{\partial f}{\partial \phi}\right)_{r,\theta} = -\,r\sin(\theta)\sin(\phi)$$

$$\left(\frac{\partial g}{\partial r}\right)_{\theta,\phi} = \sin(\theta)\sin(\phi)\ ,\ \left(\frac{\partial g}{\partial \theta}\right)_{r,\phi} = r\cos(\theta)\sin(\phi)\ ,\ \left(\frac{\partial g}{\partial \phi}\right)_{r,\theta} = r\sin(\theta)\cos(\phi)$$

$$\left(\frac{\partial h}{\partial r}\right)_{\theta,\phi} = \cos(\theta)\ ,\ \left(\frac{\partial h}{\partial \theta}\right)_{r,\phi} = -\,r\sin(\theta)\ ,\ \left(\frac{\partial h}{\partial \phi}\right)_{r,\theta} = 0$$

Thus, $|J| = \sin(\theta)\cos(\phi)[r^2 \sin^2(\theta)\cos(\phi)]$

- $r \cos(\theta)\cos(\phi)$ $[-r \sin(\theta)\cos(\theta)\cos(\phi)]$
- $r \sin(\theta)\sin(\phi)$ $[-r \sin^2(\theta)\sin(\phi) - r \cos^2(\theta)\sin(\phi)]$

Using the relationship
$\cos^2(\theta) + \sin^2(\theta) = 1$.
The Jacobian can be simplified:

$$|J| = r^2 \sin^2(\theta)\cos^2(\phi)\sin(\theta) + r^2 \cos^2(\theta)\cos^2(\phi)\sin(\theta) + r^2 \sin(\theta)\sin^2(\phi)$$
$$= r^2 \sin(\theta)\{\sin^2(\theta)\cos^2(\phi) + \cos^2(\theta)\cos^2(\phi) + \sin^2(\phi)\}$$
$$= r^2 \sin(\theta)$$

In spherical polar coordinates:
$dV = r^2 \sin(\theta)\, dr\, d\theta\, d\phi$
or
$dx\, dy\, dz = r^2 \sin(\theta)\, dr\, d\theta\, d\phi$
and

$$Volume = \iiint G(r,\theta,\phi)\, r^2 dr\, \sin(\theta)\, d\theta\, d\phi$$

where the limits can extend from $r = 0$ to ∞, $\theta = 0$ to π and $\phi = 0$ to 2π.